INTRODUCTION TO CALCULUS

Precalculus

Chapter 12

- This Slideshow was developed to accompany the textbook
  - *Precalculus*
  - *By Richard Wright*
  - *https://www.andrews.edu/~rwright/Precalculus-RLW/Text/TOC.html*
- Some examples and diagrams are taken from the textbook.

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12-01 INTRODUCTION TO LIMITS

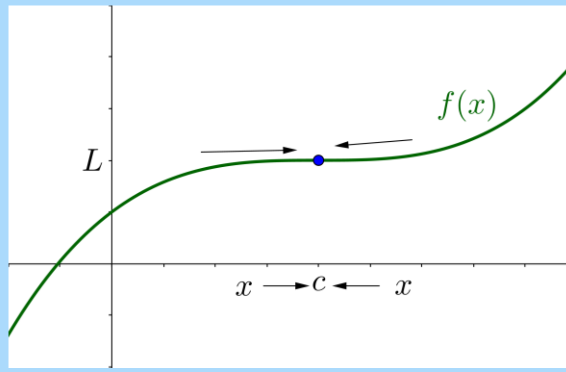
In this section, you will:

- Evaluate limits with a table.
- Identify when limits fail to exist.
- Evaluate limits by substitution.

12-01 INTRODUCTION TO LIMITS

- Limit
  - If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L.

$$\lim_{x \to c} f(x) = L$$



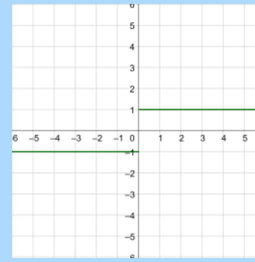
12-01 INTRODUCTION TO LIMITS

- Ways to find limits
  - Estimate Numerically (Table)

$$\lim_{x \to -2} \frac{x^2 + 4x + 4}{x + 2} = 0$$

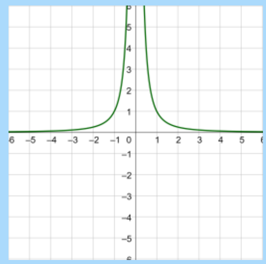
x	$f(x)$
-2.1	-0.1
-2.01	-0.01
-2.001	-0.001
-2	
-1.999	0.001
-1.99	0.01
-1.9	0.1

12-01 INTRODUCTION TO LIMITS



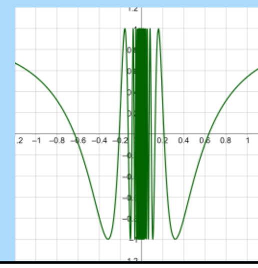
- Limits that fail to exist

1. $f(x)$ approaches different numbers from both sides



2. $f(x)$ increases of decreases without bound

3. $f(x)$ oscillates between 2 fixed values
   - $\cos\left(\frac{1}{x}\right)$



12-01 INTRODUCTION TO LIMITS

- Properties of Limits

$$\lim_{x \to c} b = b$$

$$\lim_{x \to c} x = c$$

$$\lim_{x \to c} x^n = c^n$$

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

- Let $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$

$$\lim_{x \to c} bf(x) = bL$$

$$\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$$

$$\lim_{x \to c} f(x)g(x) = LK$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$$

$$\lim_{x \to c} [f(x)]^n = L^n$$

12-01 INTRODUCTION TO LIMITS

- Evaluate
- $\lim_{x \to 2} 3x^2$
- $\lim_{x \to 1} (4x^3 - 2x^2 + 17)$
- $\lim_{x \to 2} \frac{x^2 - 4}{x}$

$$= 3(2)^2$$
$$= 12$$

$$= 4(1)^3 - 2(1)^2 + 17$$
$$= 19$$

$$= \frac{(2)^2 - 4}{2}$$
$$= \frac{0}{2}$$
$$= 0$$

12-02 EVALUATING LIMITS

In this section, you will:

- Evaluate limits that are indeterminant.
- Evaluate one-sided limits.
- Evaluate a limit from calculus.

12-02 EVALUATING LIMITS

- What if direct substitution give

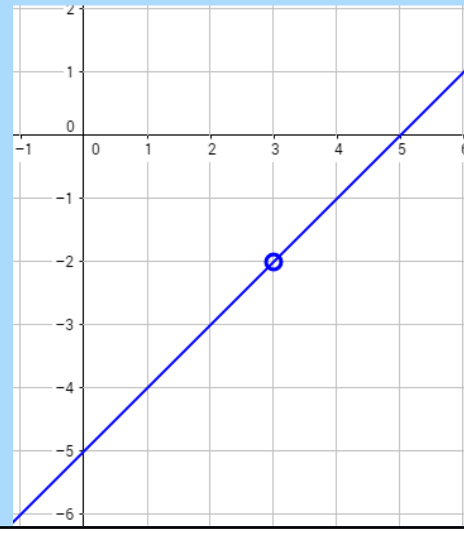
$$\lim_{x \to c} f(x) = \frac{0}{0}$$

- called indeterminant form

- Dividing out technique
  - Factor
  - Cancel common factors
  - Then find the limit

12-02 EVALUATING LIMITS

- Evaluate
- $\lim_{x \to 3} \frac{x^2 - 8x + 15}{x - 3}$



$$= \frac{(3)^2 - 8(3) + 15}{(3) - 3} = \frac{0}{0}$$

$$\lim_{x \to 3} \frac{(x-3)(x-5)}{(x-3)}$$

$$\lim_{x \to 3} (x-5)$$

$$= 3 - 5$$

$$= -2$$

12-02 EVALUATING LIMITS

- Rationalizing Technique
  - Get radicals out of numerator
  - Multiply by conjugate of numerator

- Evaluate
- $\lim_{x \to 0} \frac{\sqrt{x+9}-3}{x}$

$$= \frac{\sqrt{0+9}-3}{0} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\left(\sqrt{x+9}-3\right)\left(\sqrt{x+9}+3\right)}{x\left(\sqrt{x+9}+3\right)}$$

$$\lim_{x \to 0} \frac{x+9-9}{x\left(\sqrt{x+9}+3\right)}$$

$$\lim_{x \to 0} \frac{x}{x\left(\sqrt{x+9}+3\right)}$$

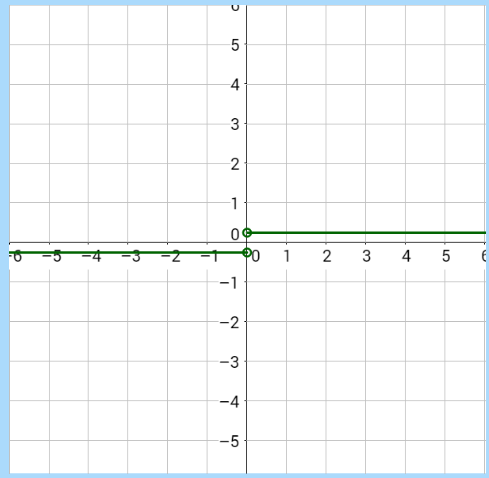
$$\lim_{x \to 0} \frac{1}{\sqrt{x+9}+3}$$

$$= \frac{1}{\sqrt{0+9}+3}$$

$$= \frac{1}{6}$$

12-02 EVALUATING LIMITS

- One-sided Limits
  - Limit found from only one direction
  - $\lim_{x \to c^-} f(x)$ - from left
  - $\lim_{x \to c^+} f(x)$ - from right



- Evaluate
  - $\lim_{x \to 0^-} \frac{|x|}{4x}$
  - $\lim_{x \to 0^+} \frac{|x|}{4x}$

$$= -\frac{1}{4}$$

$$= \frac{1}{4}$$

12-02 EVALUATING LIMITS

- A limit from calculus

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

- Always gives indeterminant case
- For the function $f(x)=2x^2+1$ find

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$\lim_{h\to 0}\frac{(2(2+h)^2+1)-(2(2)^2+1)}{h}$$

$$\lim_{h\to 0}\frac{(2(4+4h+h^2)+1)-(2(4)+1)}{h}$$

$$\lim_{h\to 0}\frac{8+8h+2h^2+1-9}{h}$$

$$\lim_{h\to 0}\frac{8h+2h^2}{h}$$

$$\lim_{h\to 0}\frac{h(8+2h)}{h}$$

$$\lim_{h\to 0} 8+2h$$

$$=8+2(0)$$

$$=8$$

12-03 DERIVATIVES

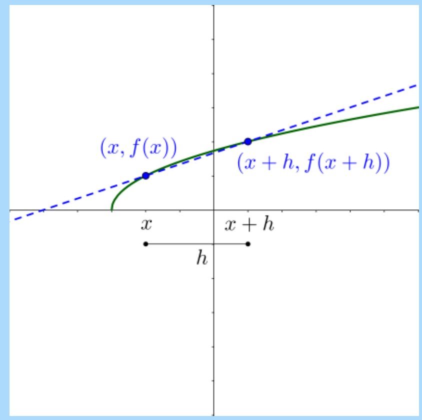
In this section, you will:

- Find the derivative of a function.
- Find the slope of the tangent line to a function.

12-03 DERIVATIVES

- Calculus is based on two main problems
  - Finding the slope of the tangent line to a function (finding rate of change)
  - Find area
- $Slope = m = \frac{y_2-y_1}{x_2-x_1}$
- $= \frac{f(x+h)-f(x)}{(x+h)-x}$
- $= \frac{f(x+h)-f(x)}{h}$
- Let h become very small
- *Slope at a point*

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$



12-03 DERIVATIVES

- Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Gives a function for slope, or rate of change, of a function

12-03 DERIVATIVES

- Find the slope of $f(x) = x^3$ at $(2, 8)$
- Find the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$
$$f'(x) = \lim_{h \to 0} (3x^2 + 3xh + h^2)$$
$$f'(x) = 3x^2 + 3x(0) + (0)^2$$
$$f'(x) = 3x^2$$

Plug in the point $(2, 8)$ or $x = 2$

$$f'(2) = 3(2)^2$$
$$f'(2) = 12$$

12-03 DERIVATIVES

- Find the derivative of $f(x) = x^2 - 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\left((x+h)^2 - 2\right) - (x^2 - 2)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - 2) - (x^2 - 2)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$f'(x) = \lim_{h \to 0} (2x + h)$$

$$f'(x) = 2x + (0) = 2x$$

12-03 DERIVATIVES

- Find the derivative of $f(x) = \sqrt{x} + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\left(\sqrt{x+h} + 1\right) - \left(\sqrt{x} + 1\right)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

12-04 LIMITS AT INFINITY AND LIMITS OF SEQUENCES

In this section, you will:

- Evaluate limits at infinity.
- Determine whether sequences converge.
- Evaluate limits of sequences.

12-04 LIMITS AT INFINITY AND LIMITS OF SEQUENCES

- Limits at Infinity
- $\lim_{x\to\infty} \frac{1}{x^r} = 0$
- $\lim_{x\to-\infty} \frac{1}{x^r} = 0$

- Evaluate
- $\lim_{x\to\infty} \frac{1+5x-3x^3}{x^3}$

$$\lim_{x\to\infty}\left(\frac{1}{x^3}+\frac{5x}{x^3}-\frac{3x^3}{x^3}\right)$$
$$\lim_{x\to\infty}\left(\frac{1}{x^3}+\frac{5}{x^2}-3\right)$$
$$= 0 + 0 - 3$$
$$= -3$$

12-04 LIMITS AT INFINITY AND LIMITS OF SEQUENCES

- Shortcut
- N = degree of numerator
- D = degree of denominator
- N < D → 0
- N = D → leading coefficients
- N > D → No limit

- Evaluate
- $\lim_{x\to\infty} \frac{-x+4}{5x^2+2}$
- $\lim_{x\to\infty} \frac{-x^2+4}{5x^2+2}$

$N = 1, D = 2$

$N < D$ → 0

$$\lim_{x\to\infty} \frac{-x+4}{5x^2+2} = 0$$

$N = 2, D = 2$

$N = D$ → leading coefficients

$$\lim_{x\to\infty} \frac{-1x^2+4}{5x^2+2} = -\frac{1}{5}$$

12-04 LIMITS AT INFINITY AND LIMITS OF SEQUENCES

- Limits of Sequences
    - If terms of a sequence approach a value as $n \to \infty$, then it converges.
    - Otherwise, it diverges.

12-04 LIMITS AT INFINITY AND LIMITS OF SEQUENCES

- Find the limit of the sequence
- $a_n = \frac{(n-3)(4n-1)}{4-3n-n^2}$

$$\lim_{n\to\infty} \frac{(n-3)(4n-1)}{4-3n-n^2}$$

$$\lim_{n\to\infty} \frac{4n^2-13n+3}{-n^2-3n+4}$$

$$N = 2, D = 2$$

$N = D$ → leading coefficients

$$= \frac{4}{-1} = -4$$

12-04 LIMITS AT INFINITY AND LIMITS OF SEQUENCES

- Find the limit of $a_n = \frac{5}{n^3} \cdot \left[\frac{n(n+1)(2n+1)}{6}\right]$.

$$\lim_{n\to\infty} \frac{5}{n^3} \cdot \left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{5}{n^3} \cdot \left[\frac{2n^3+3n^2+n}{6}\right]$$

$$\lim_{n\to\infty} \frac{10n^3+15n^2+5n}{6n^3}$$

$$= \frac{10}{6} = \frac{5}{3}$$

12-05 INTEGRALS

In this section, you will:

- Find the limit of sums as n approaches ∞.
- Evaluate an integral.

12-05 INTEGRALS

- Properties of Sums

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4+2n^3+n^2}{4}$$

- Associative Property

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

- Distributive Property (Factoring)

$$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$$

12-05 INTEGRALS

- Find the limit of $S_n = \sum_{i=1}^{n} \frac{i-5}{n^2}$ as $n \to \infty$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i-5}{n^2}$$

Split into two fractions

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n^2} - \frac{5}{n^2}\right)$$

Apply associative property

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{i}{n^2} - \sum_{i=1}^{n} \frac{5}{n^2}\right)$$

i is the index for summation. Factor out everything that is not i.

$$\lim_{n\to\infty} \left(\frac{1}{n^2} \sum_{i=1}^{n} i - \frac{5}{n^2} \sum_{i=1}^{n} 1\right)$$

Apply the sum formulas

$$\lim_{n\to\infty} \left(\frac{1}{n^2} \cdot \frac{n^2+n}{2} - \frac{5}{n^2} \cdot n\right)$$

$$\lim_{n\to\infty} \left(\frac{n^2+n}{2n^2} - \frac{5n}{n^2}\right)$$

Split into two limits

Evaluate

$$\lim_{n\to\infty}\frac{n^2+n}{2n^2}-\lim_{n\to\infty}\frac{5n}{n^2}$$

$$\frac{1}{2}-0$$

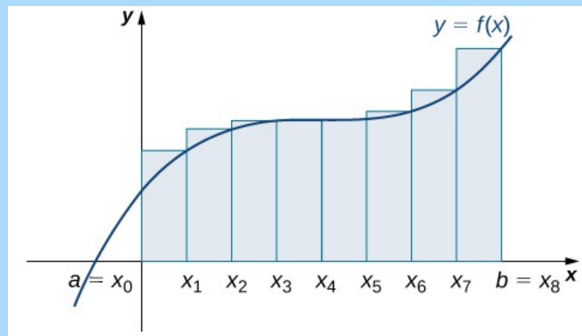
$$=\frac{1}{2}$$

12-05 INTEGRALS

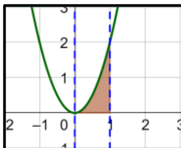
- The Area Problem
  - Find the area between the graph and the x-axis between two x-values a and b
  - Imagine drawing a bunch of rectangles with their top corner on the graph
  - The area would approximately be the area of the rectangles
  - The more rectangles, the better the approximation. Let there be ∞ rectangles

$$Area = \sum_{i=1}^{n} \ell w$$

$$Area = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right)\left(\frac{b-a}{n}\right)$$



n is the number of rectangles



12-05 INTEGRALS

- Find the area bounded by $f(x) = 2x^2$ and $x = 0$ and $x = 1$

$$\int_0^1 2x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right)\left(\frac{b-a}{n}\right)$$

Plug in a and b

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(0 + \frac{1-0}{n} i\right)\left(\frac{1-0}{n}\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)$$

Plug in $f(x)$

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i^2}{n^3}$$

Factor out non i's

$$\lim_{n\to\infty}\left(\frac{2}{n^3}\right)\sum_{i=1}^{n} i^2$$

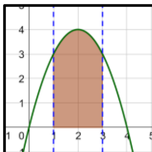
Apply sum formula

$$\lim_{n\to\infty}\left(\frac{2}{n^3}\right)\left(\frac{2n^3+3n^2+n}{6}\right)$$

$$\lim_{n\to\infty}\left(\frac{4n^3+6n^2+2n}{6n^3}\right)$$

Evaluate

$$=\frac{4}{6}=\frac{2}{3}$$



12-05 INTEGRALS

- Find the area bounded by $f(x) = 4x - x^2$ and $x = 1$ to $x = 3$

$$\int_1^3 4x - x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a + \frac{b-a}{n}i\right)\left(\frac{b-a}{n}\right)$$

Plug in a and b

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(1 + \frac{3-1}{n}i\right)\left(\frac{3-1}{n}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

Plug in $f(x)$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(4\left(1+\frac{2i}{n}\right) - \left(1+\frac{2i}{n}\right)^2\right)\left(\frac{2}{n}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(4+\frac{8i}{n} - \left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)\right)\left(\frac{2}{n}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(3+\frac{4i}{n}-\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{6}{n}+\frac{8i}{n^2}-\frac{8i^2}{n^3}\right)$$

Apply the Associative Property and Factor non i's

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{6}{n}+\sum_{i=1}^{n}\frac{8i}{n^2}-\sum_{i=1}^{n}\frac{8i^2}{n^3}\right)$$

$$\lim_{n\to\infty}\left(\frac{6}{n}\sum_{i=1}^{n}1+\frac{8}{n^2}\sum_{i=1}^{n}i-\frac{8}{n^3}\sum_{i=1}^{n}i^2\right)$$

12-05 INTEGRALS

$$\lim_{n\to\infty}\left(\frac{6}{n}\sum_{i=1}^{n}1+\frac{8}{n^2}\sum_{i=1}^{n}i-\frac{8}{n^3}\sum_{i=1}^{n}i^2\right)$$

Apply the sum formulas

$$\lim_{n\to\infty}\left(\frac{6}{n}(n)+\frac{8}{n^2}\left(\frac{n^2+n}{2}\right)-\frac{8}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)\right)$$

$$\lim_{n\to\infty}\left(6+\left(\frac{8n^2+8n}{2n^2}\right)-\left(\frac{16n^3+24n^2+8n}{6n^3}\right)\right)$$

Separate into individual limits

$$\lim_{n\to\infty}6+\lim_{n\to\infty}\left(\frac{8n^2+8n}{2n^2}\right)-\lim_{n\to\infty}\left(\frac{16n^3+24n^2+8n}{6n^3}\right)$$

Evaluate

$$=6+\frac{8}{2}-\frac{16}{6}$$

$$=6+4-\frac{8}{3}$$

$$=\frac{22}{3}$$